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Technical Note

Forced convection in a bi-disperse porous medium channel: a conjugate problem

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Abstract

Forced convection in a plane channel filled with a saturated bi-disperse porous medium, coupled with conduction in plane slabs bounding the channel, is investigated analytically on the basis of a two-velocity, two-temperature model. It is found that the effect of the finite thermal resistance due to the slabs is to reduce both the heat transfer to the porous medium and the degree of local thermal non-equilibrium. An increase in value of the Péclet number leads to a decrease in the rate of exponential decay in the downstream direction but does not affect the value of a suitably defined Nusselt number. The dependence of Nusselt number on Biot number associated with the boundary slabs, the interphase heat exchange parameter, the interphase thermal conductivity ratio, the interphase effective permeability ratio, and the macroscopic void fraction, is investigated.

1. Introduction

There has recently been renewed interest in the problem of forced convection in a porous medium channel because of the use of hyperporous media in the cooling of electronic equipment. Recent surveys have been made by Nield and Bejan [1] and by Lauriat and Ghafir [2]. With two exceptions, in each case a regular porous medium was considered. The exceptions are the studies by Nield and Kuznetsov [3] and Kuznetsov and Nield [4] of flow in a channel occupied by a bi-dispersed porous medium bounded by parallel plates. In the first paper fully developed convection was studied for the case of uniform temperature or uniform heat flux boundaries. In the second paper thermally developing convection for uniform temperature boundaries was treated.

A bi-dispersed porous medium (BDPM), as defined in [5], is composed of clusters of large particles that are agglomerations of small particles (Fig. 1a). Thus there are macro-pores between the clusters and micro-pores within them. A BDPM may thus be looked at as a standard porous medium in which the solid phase is replaced by another porous medium, whose temperature may be denoted by T_p if local thermal equilibrium is assumed within each cluster. We can then talk about the f-phase (the macro-pores) and the p-phase (the remainder of the structure). An alternative way of looking at the structure is to regard it as a porous medium in which fractures or tunnels have been introduced. One

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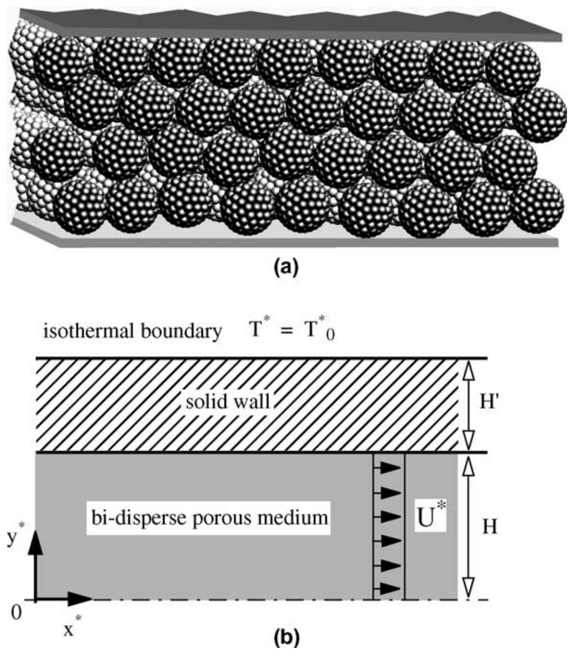

Fig. 1. (a) Sketch of a bi-disperse porous medium. (b) Definition sketch.

can then think of the f-phase as being a 'fracture phase' and the p-phase as being a 'porous phase'.

Nield and Kuznetsov [3] modeled the steady-state heat transfer by a pair of coupled equations for T_f and T_p (see Eqs. (8) and (9)). They also introduced a coupled pair of momentum equations involving the velocities in the two phases, and showed that the effect of that coupling was simply to modify the permeabilities in the two phases.

In this paper the analysis presented in [3] is extended to a conjugate problem, involving the coupling of convection in the channel with conduction in adjacent solid slabs. The analysis is similar to that employed by Nield and Kuznetsov [6], and the present paper may be regarded as an extension of [6] to a two velocity (as well as two temperature) model.

2. Analysis

The geometry of the problem is illustrated in Fig. 1b. We consider a porous medium channel of half width H bounded on each side by a boundary solid slab of thickness H'. We consider the case where the outside of the boundary slabs is maintained at a uniform constant temperature T_0^* and we neglect axial conduction both in the bi-disperse porous medium and in the boundary slabs. The neglect of axial conduction within the porous medium is justified if the Péclet number is sufficiently large. The neglect of conduction in the slabs is consistent with the neglect of axial conduction in the porous medium, together with the uniform temperature imposed on the outside.

When the axial heat flux is zero, the temperature T'^* in the solid slabs is independent of the axial coordinate x^*. (We use asterisks to denote dimensional variables.) If T_0^* is the constant outside temperature, that at $y^* = H + H'$, then the solution of the heat conduction equation is

$$T'^* = T_0^* + \beta(H + H' - y^*), \tag{1}$$

where β is constant. The temperature at the channel wall, at $y^* = H$, is thus $T_w^* = T_0^* + \beta H'$, and the wall heat flux is $k'\beta$, where k' is the slab conductivity. Let T_f^*, T_p^*, be respectively the temperature in the f-, p- phase of the porous medium, and ϕ be the macropore volume fraction. Equating REV averages of the temperature and heat flux to the wall values we have, at $y^* = H$,

$$\phi T_f^* + (1-\phi)T_p^* = T_0^* + \beta H', \tag{2}$$

$$\phi k_f(\partial T_f^*/\partial y^*) + (1-\phi)k_p(\partial T_p^*/\partial y^*) = -k'\beta, \tag{3}$$

where k_f and k_p are the f-phase and p- phase conductivities, respectively.

Writing

$$\theta_f^* = T_f^* - T_0^*, \quad \theta_p^* = T_p^* - T_0^*, \tag{4}$$

and eliminating β, we have the boundary condition

$$\phi k_f(\partial\theta_f^*/\partial y^*) + (1-\phi)k_p(\partial\theta_p^*/\partial y^*) = -(k'/H')(\phi\theta_f^* + (1-\phi)\theta_p^*), \quad \text{at } y^* = H. \tag{5}$$

Because the differential equations system, Eqs. (10) and (11) below, is of fourth order, we need *two* boundary conditions at $y^* = H$. Following Nield and Kuznetsov [6], we postulate a *uniformity principle* that requires that the boundary condition holds for all values of the macroscopic fluid volume fraction ϕ. Accordingly, we have

$$k_f(\partial\theta_f^*/\partial y^*) = -(k'/H')\theta_f^* \quad \text{and}$$
$$k_p(\partial\theta_p^*/\partial y^*) = -(k'/H')\theta_p^* \quad \text{at } y^* = H. \tag{6}$$

We also have the symmetry conditions

$$\partial\theta_f^*/\partial y^* = 0, \ \partial\theta_p^*/\partial y^* = 0 \quad \text{at } y^* = 0. \tag{7}$$

We assume that T_p^* and T_f^* are governed by the steady state heat transfer (energy) equations

$$\phi\nabla\cdot(k_f\nabla T_f^*) + h_{fp}(T_p^* - T_f^*) = \rho c_P \mathbf{v}_f^*\cdot\nabla T_f^* \tag{8}$$

$$(1-\phi)\nabla\cdot(k_p\nabla T_p^*) + h_{fp}(T_f^* - T_p^*) = \rho c_P \mathbf{v}_p^*\cdot\nabla T_p^* \tag{9}$$

Here ϕ is the volume fraction of the f-phase, $\mathbf{v}_f^*$ and $\mathbf{v}_p^*$ are the Darcy velocities in the two phases, k_f and k_p are the effective thermal conductivities of the two phases (since the f-phase is entirely fluid that means that k_f is actually the thermal conductivity of the fluid), ρc_P is the heat capacity per unit volume of the fluid, and h_{fp} is the coefficient for heat transfer between the two phases (with the specific area incorporated into the coefficient).

The reader will note that the addition of Eqs. (8) and (9) produces an equation which is the standard thermal energy equation based on an REV approach.

We now consider the case where the Darcy velocity is axial and has the uniform values U_f^*, U_p^* in the respective f-, p-phases.

We write

$$x = x^*/PeH, \quad y = y^*/H, \quad \theta_f = \theta_f^*/T_{ref}^*,$$
$$\theta_p = \theta_p^*/T_{ref}^*, \tag{10}$$

$$\hat{u}_f = u_f^*/U_m^*, \quad \hat{u}_p = u_p^*/U_m^*, \quad U_r = U_p^*/U_f^*. \tag{11}$$

where T_{ref}^* is any convenient temperature scale and

$$U_m^* = \phi U_f^* + (1-\phi)U_p^*, \tag{12}$$

while Pe is the Péclet number defined as

$$Pe = U_m^* H(\rho c_P)_f / k_{eff}. \tag{13}$$

It will be noted that that U_m^* is equal to the volumetric flux density through the BDPM (an overall Darcy number) and is thus an easily measured physical quantity.

It will also be noted that Pe enters into the scaling of the axial coordinate, but plays no further role.

For convenience, we perform the subsequent algebra in terms of the parameters

$$N_f = \frac{\phi}{\phi + (1-\phi)k_r}, \quad N_p = \frac{(1-\phi)k_r}{\phi + (1-\phi)k_r}, \quad N_h = \eta, \tag{14}$$

where η is the interface heat exchange parameter and k_r is the conductivity defined by

$$\eta = h_{fp}H^2/k_{eff}, \quad k_r = k_p/k_f,$$
$$k_{eff} = \phi k_f + (1-\phi)k_p. \tag{15}$$

Eqs. (8) and (9) now take the form

$$[N_f\partial^2/\partial y^2 - N_h - \hat{u}_f\partial/\partial x]\theta_f + N_h\theta_p = 0 \tag{16}$$

$$N_h\theta_f + [N_p\partial^2/\partial y^2 - N_h - \hat{u}_p\partial/\partial x]\theta_p = 0. \tag{17}$$

The boundary conditions (6) and (7) become

$$\partial\theta_f/\partial y + L_f\theta_f = 0, \quad \partial\theta_p/\partial y + L_p\theta_p = 0 \quad \text{at } y = 1, \tag{18}$$

$$\partial\theta_f/\partial y = 0, \quad \partial\theta_p/\partial y = 0 \quad \text{at } y = 0. \tag{19}$$

Here L_f and L_p are defined by

$$L_f = Bi[\phi + (1-\phi)k_r], \quad L_p = Bi[\phi + (1-\phi)k_r]/k_r, \tag{20}$$

where in turn the Biot number Bi is defined as

$$Bi = k'H/k_{eff}H'. \tag{21}$$

The homogeneous system of equations (16)–(19) can be solved using the method of separation of variables. Letting

$$\theta_f = \Theta_f(y)e^{\lambda x}, \quad \theta_p = \Theta_p(y)e^{\lambda x} \tag{22}$$

and denoting d/dy by D, we get

$$(N_fD^2 - N_h - \hat{u}_f\lambda)\Theta_f + N_h\Theta_p = 0, \tag{23}$$

$$(N_pD^2 - N_h - \hat{u}_p\lambda)\Theta_p + N_h\Theta_f = 0, \tag{24}$$

$$D\Theta_f + L_f\Theta_f = 0, \quad D\Theta_p + L_p\Theta_p = 0 \quad \text{at } y = 1, \tag{25}$$

$$D\Theta_f = 0, \quad D\Theta_p = 0 \quad \text{at } y = 0. \tag{26}$$

Eliminating Θ_p, we get

$$\{(N_fD^2 - N_h - \hat{u}_f\lambda)(N_pD^2 - N_h - \hat{u}_p\lambda) - N_h^2\}\Theta_f = 0 \tag{27}$$

$$D\Theta_f + L_f\Theta_f = 0,$$
$$(D + L_p)(N_fD^2 - N_h - \hat{u}_f\lambda)\Theta_f = 0 \quad \text{at } y = 1. \tag{28}$$

The solution of Eq. (27) subject to the symmetry requirement (26) is

$$\Theta_f = A\cos s_1y + B\cosh s_2y, \tag{29}$$

where A and B are constants and

$$as_1^4 + bs_1^2 + c = 0, \quad as_2^4 - bs_2^2 + c = 0 \tag{30}$$

where

$$a = N_fN_p, \tag{31}$$

$$b = N_h(N_f + N_p) + (N_f\hat{u}_p + N_p\hat{u}_f)\lambda, \tag{32}$$

$$c = N_h(\hat{u}_f + \hat{u}_p)\lambda + \hat{u}_f\hat{u}_p\lambda^2. \tag{33}$$

We require that s_1 and s_2 be real and positive and λ be real and negative. Thus

$$s_1 = \{[-b + (b^2 - 4ac)^{1/2}]/2a\}^{1/2}, \tag{34}$$

$$s_2 = \{[b + (b^2 - 4ac)^{1/2}]/2a\}^{1/2}, \tag{35}$$

Substituting into Eq. (27), and eliminating A and B, we get the eigenvalue equation for λ, which can be written in the form

$$(N_fs_1^2 + N_h + \hat{u}_f\lambda)(s_1\tan s_1 - L_p)(s_2\tanh s_2 + L_f)$$
$$= (-N_fs_2^2 + N_h + \hat{u}_f\lambda)(s_2\tanh s_2 + L_p)(s_1\tan s_1 - L_f) \tag{36}$$

The significant eigenvalue is the negative root of smallest magnitude. (The set of equations (31)–(36) can be solved iteratively to compute this eigenvalue.) The corresponding values of s_1 and s_2 are real. The eigenvector is obtained from

$$B/A = (s_1 \sin s_1 - L_f \cos s_1)/(s_2 \sinh s_2 + L_f \cosh s_2), \quad (37)$$

and from Eq. (23) one obtains

$$\Theta_p = N_h^{-1}\{(N_f s_1^2 + N_h + \lambda)A \cos s_1 y + (-N_f s_2^2 + N_h + \lambda)B \cosh s_2 y\}. \quad (38)$$

With the temperature distribution completely found, one can then compute the heat transfer. Matching the heat flux at the channel wall gives

$$q'' = \phi k_f (\partial T_f^*/\partial y^*)_{y^*=H} + (1-\phi)k_p(\partial T_p^*/\partial y^*)_{y^*=H} \quad (39)$$

The Nusselt number is defined by

$$Nu = 2Hh/k_{\text{eff}}, \quad (40)$$

where, in turn,

$$h = q''/(T_w^* - T_{b,m}^*) \quad (41)$$

where the mean bulk temperature is defined by

$$T_{b,m}^* = \frac{1}{U_m^* H}\int_0^H u^*\{\phi T_f^* + (1-\phi)T_p^*\}\,dy^* = \frac{1}{H}\int_0^H \{\phi \hat{u}_f T_f^* + (1-\phi)\hat{u}_p T_p^*\}\,dy^*. \quad (42)$$

The calculation of Nu is now straightforward. The factor $T_{\text{ref}}\,e^{\lambda x}$ that appears in both q'' and $T_{b,m}^* - T_0^*$ now cancels in the calculation. One can use Eq. (37) and the relations

$$k_{\text{eff}} = \phi k_f + (1-\phi)k_p = U_m^* H(\rho c_P)_f(N_f + N_p), \quad (43)$$

$$\phi k_f = U_m^* H(\rho c_P)_f N_f, \quad (44)$$

$$(1-\phi)k_p = U_m^* H(\rho c_P)_f N_p, \quad (45)$$

to obtain the formula for the Nusselt number in the form,

$$Nu = \frac{(L_f C_2 + s_2 S_2)s_1 S_1[N_f + N_p M_1] + (L_f C_1 - s_1 S_1)s_2 S_2[N_f + N_p M_2]}{\left(\frac{N_f + N_p}{2}\right)\left\{\frac{(L_f C_2 + s_2 S_2)S_1}{s_1}\left[\phi\hat{u}_f + (1-\phi)\hat{u}_p M_1\right] - \frac{(L_f C_1 - s_1 S_1)S_2}{s_2}\left[\phi\hat{u}_f + (1-\phi)\hat{u}_p M_2\right]\right\}}, \quad (46)$$

where $C_1 = \cos s_1$, $C_2 = \cosh s_2$, $S_1 = \sin s_1$, $S_2 = \sinh s_2$, and

$$M_1 = (N_f s_1^2 + N_h + \lambda)/N_h,$$
$$M_2 = (-N_f s_2^2 + N_h + \lambda)/N_h. \quad (47)$$

The solution for the case $k_f = k_p$ and $U_f^* = U_p^*$ is especially simple. One finds that in this case

$$\Theta_f = A \cos \mu y, \quad (48)$$

and

$$Nu = 2\mu^2, \quad (49)$$

Table 1
Values of Nusselt number versus Biot number, calculated from Eqs. (49) and (50), for the case of thermal equilibrium (reproduced from [6])

Bi	μ	*Nu*
0	0	0
0.01	0.09983	0.01993
0.05	0.22176	0.09835
0.1	0.31105	0.1935
0.5	0.65327	0.8535
1	0.86033	1.4803
5	1.31384	3.4524
10	1.42887	4.0833
50	1.54001	4.7433
100	1.55525	4.8376
500	1.56766	4.9151
1000	1.56923	4.9250
∞	1.57080 ($\pi/2$)	4.9348 ($\pi^2/2$)

where μ is the smallest positive root of the equation

$$x \tan x = Bi. \quad (50)$$

Some results based on equations (49) and (50) are given in Table 1.

For the general case, we present our results in terms of the independent parameters ϕ, Bi, η, k_r, and U_r. To this end, we employ the expressions

$$\hat{u}_f = 1/[\phi + (1-\phi)U_r], \quad (51)$$

$$\hat{u}_p = U_r/[\phi + (1-\phi)U_r]. \quad (52)$$

The eigenfunction solution so far obtained for our parabolic differential equation system contains a multiplicative factor whose determination requires that an upstream ("initial") condition be specified. The eigenfunction is such that its shape (expressed by the dependence on y) does not evolve with distance downstream but the amplitude (expressed by the dependence on x) decays exponentially as x increases.

3. Results and discussion

We tested our computer code for the evaluation of the eigenvalue λ and then the Nusselt number Nu by comparing our results, for the case of very small ϕ and using a large value of U_r (for test purposes only), with known results reported by Nield and Kuznetsov [6], and in particular with those listed in Table 1. We can in fact regard our present paper as an extension, to a two-velocity model, of [6] with U_r, representing a ratio of effective permeabilities in the two phases, as a new parameter to be varied. Thus from [6] we already know that *Nu* invariably decreases as *Bi* increases. We also know that *Nu* increases/decreases as η decreases from large values (the case of local thermal equilibrium) to smaller values according to whether k_r is less/greater than unity. Also, we expect that *Nu* will also be generally less at intermediate values of ϕ (the f-phase volume fraction here, the porosity in [6]) than at the extreme values ϕ very small or close to unity with one exception (for small ϕ when k_r is less than unity). In general, *Nu* varies little with ϕ in the range [0.25, 0.5] relevant for a realistic bi-disperse porous medium. In a representative case we

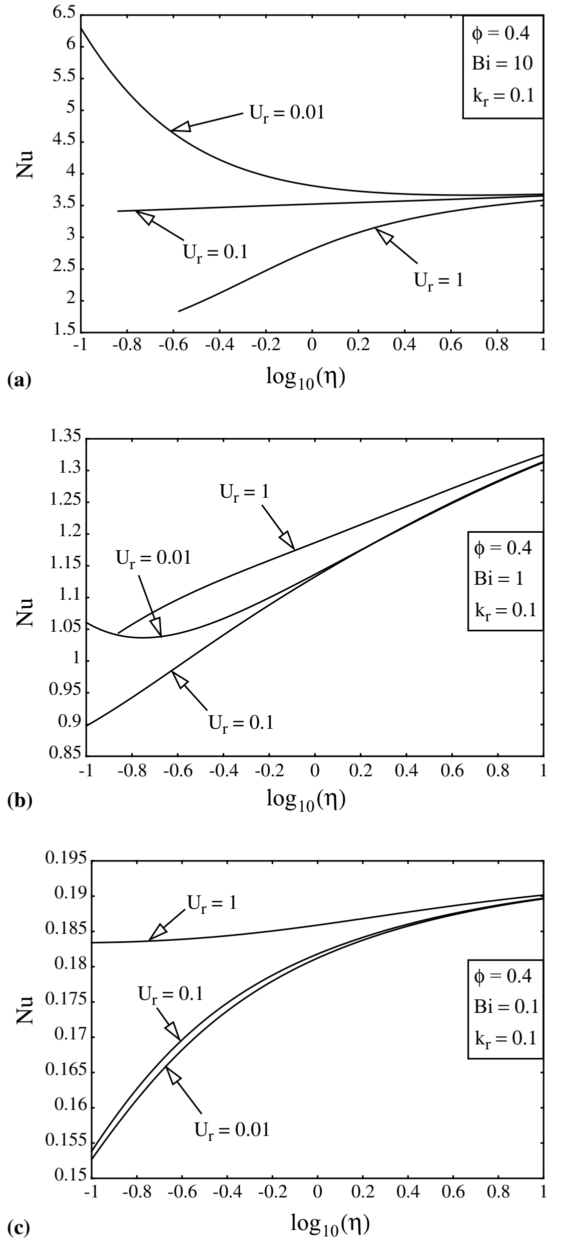

Fig. 2. Plots of Nusselt number versus interphase heat exchange parameter for various values of the velocity ratio (case $\phi = 0.4$, $k_r = 0.1$): (a) $Bi = 10$, (b) $Bi = 1$ and (c) $Bi = 0.1$.

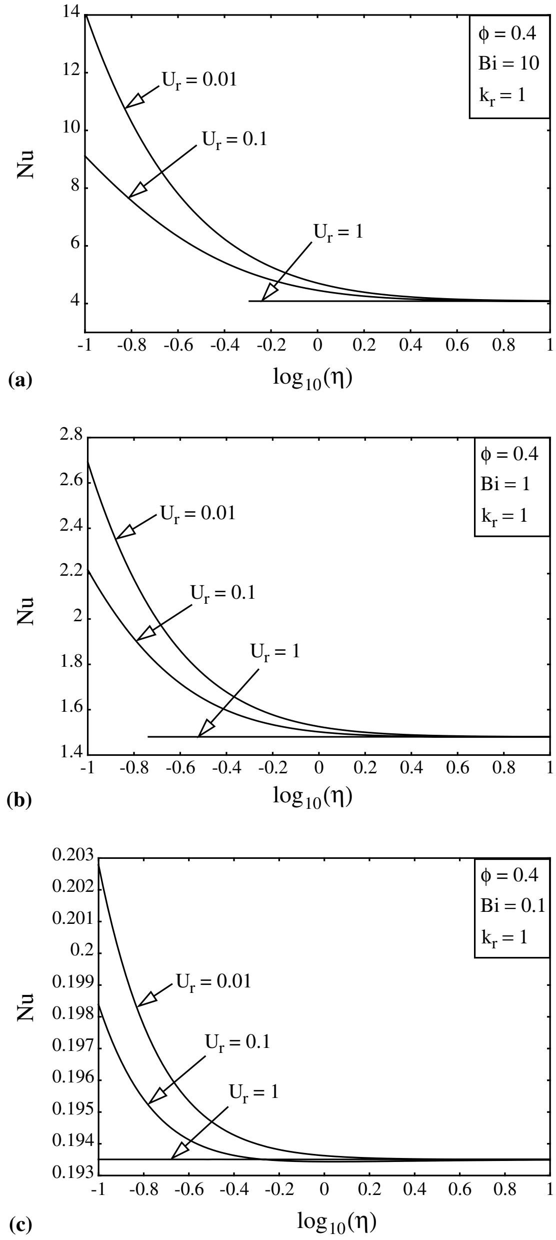

Fig. 3. Plots of Nusselt number versus interphase heat exchange parameter for various values of the velocity ratio (case $\phi = 0.4$, $k_r = 1$): (a) $Bi = 10$, (b) $Bi = 1$ and (c) $Bi = 0.1$.

found that Nu increased by only 5% as ϕ increased from 0.25 to 0.5.

Accordingly, we have chosen to present figures in each of which curves are plotted for various values of U_r, the individual curves being of Nu plotted versus η (strictly speaking log η) for a representative value (0.4) of ϕ. The figures correspond to various values of k_r ($k_r = 0.1$ in Fig. 2, $k_r = 1.0$ in Fig. 3, $k_r = 10$ in Fig. 4) and Bi (for (a) $Bi = 10$, (b) $Bi = 1$, (c) $Bi = 0.1$ in each figure). The values $U_r = 1$, 0.1 and 0.01 are used in the presentation. The first value is probably not realistic physically, and should be regarded as a limiting case. We also point out that small values of η are not physically realistic, and so we were not concerned when we ran into numerical difficulties when we attempted to decrease the value of η. (This is the reason why some of the curves presented stop at an arbitrary value of η.)

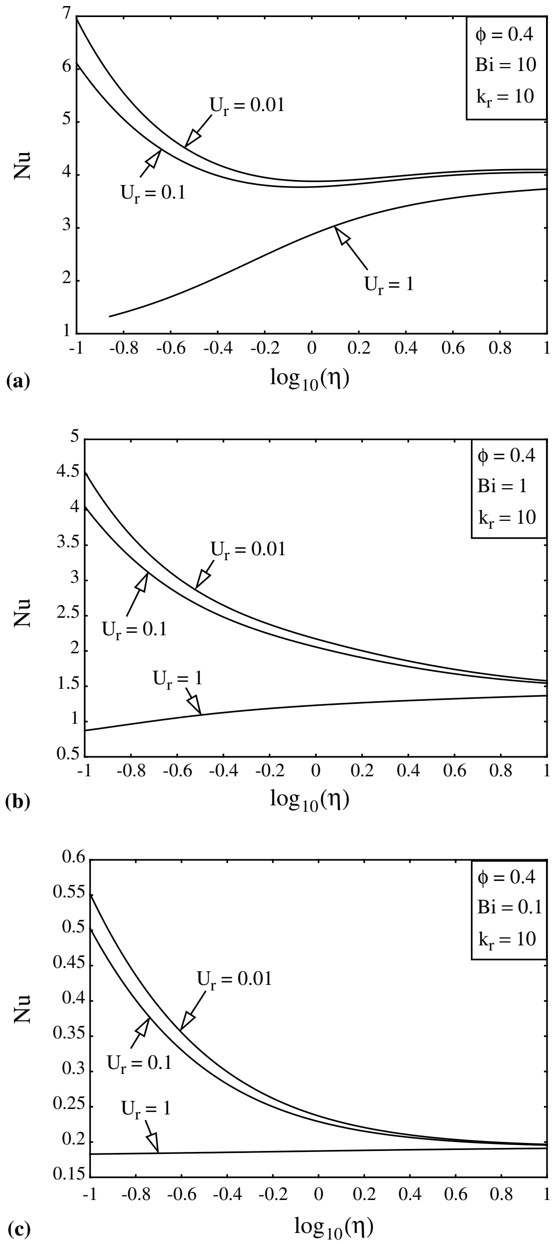

Fig. 4. Plots of Nusselt number versus interphase heat exchange parameter for various values of the velocity ratio (case $\phi = 0.4$, $k_r = 10$): (a) $Bi = 10$, (b) $Bi = 1$ and (c) $Bi = 0.1$.

The figures show an interesting variety of behavior. As expected, a major feature is the decrease in Nu as Bi decreases, and the above mentioned variation with k_r of the the direction of variation of Nu with η is generally confirmed (but Fig. 2a shows an exception). The value of Nu is especially sensitive to the value of η when k_r is small. In most cases Nu increases as U_r decreases, but the cases shown in Fig. 2b and c are exceptions. Also in most cases there is a substantial jump as U_r changes from 1.0 to 0.1, but a much smaller jump as it changes from 0.1 to 0.01 (and additional calculations showed that there was then little further change as U_r decreased further). A rather dramatic change is shown in Fig. 2a, where the direction of variation of Nu with η changes as U_r changes.

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